Absorption of sound in helium II

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(J. Exp. Theor. Phys., USSR, 20, 243, 1950).

Summary

The present paper is an attempt to construct the theory of the absorption of first and second sound in helium II. As slow processes leading to absorption of sound in helium II are considered processes of emission and absorption of elementary excitations (phonons and rotons). The temperature dependences of the absorption coefficients of first and second sound are determined. The question of the second viscosity of helium II is discussed, and expressions are obtained for the four coefficients of second viscosity of helium II.

1. Introduction.

The fact that helium II has a non-zero viscosity must in itself lead to absorption of sound propagated in helium II. However the anomalously low viscosity (*) has not permitted the discovery of the effects of absorption in helium II for sound waves of relatively low frequencies.

Only recently has it been possible to discover the absorption of ordinary sound in helium II (1). The frequency of the sound used for this purpose was about 10^8 cyc/sec. Since in helium II the specific heats C_p and C_v are practically equal, the absorption of sound connected with conductivity does not occur. Thus all the absorption observed in these experiments must be due to viscosity. However the calculation of the coefficient of viscosity from the absorption data gave values roughly an order of magnitude higher than the values from Andronikashvili's measurements (2). This discrepancy suggests that besides ordinary viscosity, helium II also has the so-called second viscosity. Moreover the processes of establishment of equilibrium in helium II must occur relatively slowly.

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* Of the order of 10^{-5} poise from 1.5 - 2°K.

As was shown previously (3), the establishment of energy equilibrium in the phonon and roton gases takes place faster than the scattering processes determining the first viscosity. As regards exchange of energy between the phonon and roton gases, this takes place mostly by scattering of phonons and rotons. The calculation of the relaxation time for processes establishing temperature equilibrium between the phonon and roton gases (in the presence of energy equilibrium in each gas separately) showed that this process is also faster than the processes of scattering of the elementary excitations responsible for ordinary viscosity. Thus the processes indicated for the establishment of energy equilibrium cannot be responsible for the anomalously large absorption of first sound in helium II. The only relatively slow processes of helium II are the processes of establishment of equilibrium with respect to numbers of phonons and rotons (compare (3). § 8). Thus we shall start out from the assumption that it is just these processes which determine the second viscosity of helium II.

The dispersion of sound in helium II, which is possible in principle, (i.e. dependence of velocity on frequency), has also not been observed up to the present time (4). It will be shown below that the sound waves used were not of sufficiently high frequency and so the dispersion was not achieved.

2. Absorption and emission of phonons and rotons.

For collisions of phonons and rotons, processes are possible as a result of which the total number of rotons and phonons can change. All such processes in helium II may be divided into three types: (a) processes in which phonons are emitted or absorbed; (b) processes in which rotons are emitted or absorbed; (c) processes in which rotons are converted into phonons or vice versa.

We shall now consider the most probable processes of each type. Only the fastest of the indicated processes will

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Emission and absorption of phonons

In the collision of two phonons, processes are possible leading to a change in the total number of phonons. The simplest of these processes – the three-phonon process – is forbidden because of the impossibility of the simultaneous fulfilment of the two conservation laws of momentum and energy. Thus we consider a 5-phonon process consisting of absorption or emission of a third phonon for collision of two phonons.

We start with the Hamiltonian function of the quantum liquid (5)

$$H = \frac{1}{2} \underline{v} \rho \underline{v} + \rho \, \varepsilon(\rho) \qquad (2.1)$$

where $\underline{v}$ and ρ are respectively the velocity and density of the liquid and $\varepsilon(\rho)$ is the energy per unit mass of the liquid.

The non-vanishing terms of the matrix elements for transitions of three phonons into two are obtained from the cubic terms in ρ' in the energy for the third approximation of the perturbation theory, from the cubic and fourth order terms in the second approximation and from the fifth order terms in the first approximation (ρ' is the deviation of the density from its value in a stationary liquid). We express the hamiltonian (2.1) in the form of a sum

$$H = H_0 + V_3 + V_4 + V_5 \qquad (2.2)$$

where H_0 is the density of sound energy containing terms quadratic in ρ', and V_3, V_4 and V_5 containing respectively the terms of third, fourth and fifth order with respect to ρ'

With the help of (2.2) in which the terms V_3, V_4 and V_5 are considered as a small perturbation, we obtain for the matrix element of the transition

$$H_{AF} = \sum_{I,II} \frac{(V_3)_{AI} (V_3)_{I,II} (V_3)_{II\,F}}{(E_A - E_I)(E_A - E_{II})} + \sum_{I} \frac{(V_4)_{AI}(V_3)_{IF}}{E_A - E_I} + (V_5)_{AF}$$

(2.3)

When dispersion is neglected, the occurrence of a denominator of the resonance type for small angles between the moments of the colliding phonons leads to divergences of the first terms in (2.3) (3). Taking account of the dispersion of the phonon part of the energy spectrum[*] i.e. of the fact that the energy is not a strictly linear function of the momentum of the phonon, eliminates the divergences of the terms indicated. It is easy to see that only those transitions will be significant which are described by the first sum in (2.3), some of the terms of which contain in their denominators products of two expressions tending to zero for neglect of the dispersion ($E_A - E_I$ and $E_A - E_{II}$). Thus we shall leave in the matrix element only the first sum depending on the terms V_3 of the third order in ρ' :

$$V_3 = \frac{v \, \rho' \, v}{2} + \frac{1}{3!} \frac{\partial}{\partial \rho}\left(\frac{c^2}{\rho}\right) \rho'^3 \qquad (2.4)$$

Suppose that in the initial state (A) we have three phonons with momenta $\underline{P_1}$, $\underline{P_2}$, $\underline{P_3}$, and in the final state (F) two phonons with momenta $\underline{P_4}$ and $\underline{P_5}$. Altogether there are possible only several tens of intermediate processes (I and II) through which the transition from state A to state F can take place. However, as has already been pointed out, the significant ones will be only those intermediate states for which at small angles between the momenta of the colliding phonons we have simultaneously[+]

$$E_A - E_I \rightarrow O \; , \; E_A - E_{II} \rightarrow O \qquad (2.5)$$

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[*] The dispersion in question here must not be confused with the dispersion of sound i.e. with the phenomenon of the dependence of the velocity of sound on frequency.

[+] It turns out that there are altogether 27 special intermediate states, but owing to the conservation laws only 15 intermediate states satisfy the condition (2.5) which particular ones depend on the relations between the magnitude of the momenta. Schematically the transitions through the 27 indicated intermediate states can be represented in the following way:

From what has been said it is absolutely clear that the calculation of probabilities of the transition considered $(3 \rightarrow 2)$

$$d\omega = \frac{2\pi}{h} \left| H_{AF} \right|^2 \delta \left(\xi_1 + \xi_2 + \xi_3 - \xi_4 - \xi_5 \right) \frac{dp_4}{(2\pi h)^3} \quad (2.6)$$

is an extremely complicated operation. Such a calculation has hardly any sense at the present time. The point is that the basic parameters of the theory of the differential coefficients of the first three orders of c the velocity of sound with respect to density and especially the dispersion parameter γ,* are known only very roughly (3). Thus we shall limit ourselves to establishing the relations between the probability of transition and the energy of collision of the phonons. This will completely determine the temperature variation of the quantities which interest us.

The non-vanishing matrix elements of the Fourier components of the density have, according to (3), the form

$$\left(\rho \right)_{n_\rho, n_\rho+1} = \sqrt{\frac{\rho \rho}{2c}} e^{-i\omega_\rho t} \sqrt{n_\rho+1} \quad ; \quad \left(\rho^* \right)_{n_\rho, n_\rho+1} = \sqrt{\frac{\rho \rho}{2c}} e^{-i\omega t} \sqrt{n_\rho} \quad ,(2.7)$$

To establish the temperature variation of any average with respect to all phonons of the probability of the process considered in (2.6), there is no need to identify the colliding phonons. Thus we may omit the indices in the momenta of the phonons. In this way we have from (2.4) and (2.7)

$$V_3 \sim p^{3/2}$$

For instance, the difference of energies $E_A - E_I$ for a concrete example when we have in the intermediate state I two phonons with momenta $p_1 + p_2$ and p_3 may be written in the form

$$\xi(p_1) + \xi(p_2) - \xi(|p_1 + p_2|) = \frac{c \, p_1 p_2}{p_1 + p_2} \left[1 - n_1 n_2 + 3\gamma (p_1 + p_2)^2 \right] (2.8)$$

where n_1 and n_2 are the unit vectors in the directions of the corresponding momenta. The matrix element of the transition will contain in its denominators the products of two expressions of the type (2.8).

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* We recall that taking account of dispersion, the dependence of the energy ξ on the momentum of the phonon has the form (see (3))

In the present context we are naturally not interested in the angular distribution of phonons. Thus we average (2.6) over the angles made by the momenta of the colliding phonons and integrate over the phase volume of the momenta of the scattered phonons. The necessary integration over angles proves to be quite simple owing to the smallness of the terms in the denominator of the expression for H_{AF} which takes account of dispersion. (Compare (3) Para. 2).

Without dwelling on the fairly simple calculations, we give the final result obtained for the probabilities averaged over angles

$$\overline{w} \sim p^{12} / (\gamma p^2)^2 \, p^5 \sim p^3 \qquad (2.9)$$

Let the total number of phonons per unit volume (not in general the equilibrium value) be N_ϕ. The rate of change of the number of phonons due to the 5-phonon process may be written in the form

$$\dot{N}_\phi = -\iiint \left[\left(n_1 n_2 n_3 (n_4+1)(n_5+1) \right) - (n_1+1)(n_2+1)(n_3+1) \, n_4 n_5 \right] d\omega \; \frac{dp_1 \, dp_2 \, dp_3}{(2\pi\hbar)^9} \qquad (2.10)$$

If the total number of phonons is not equal to the equilibrium value this means that the distribution function n contains a non-zero chemical potential μ_ϕ

$$n = \left[e^{[\varepsilon(p) - \mu_\phi]/kT} - 1 \right]^{-1}$$

For small deviations from equilibrium, the function n can be expanded in powers of μ_ϕ, and considering only the linear terms in μ_ϕ we have

$$n - n_0 = n_0 (n_0+1) \mu_\phi / kT \qquad (2.11)$$

The function n with suffix 0 corresponds to the equilibrium distribution for phonons ($\mu_\phi = 0$). After some straightforward transformations with the help of (2.11), (2.10) takes the form

$$\dot{N}_\phi = -\iiint n_{10} n_{20} n_{30} (n_{40}+1)(n_{50}+1) \, d\omega \; \frac{dp_1 \, dp_2 \, dp_3}{(2\pi\hbar)^9} \; \frac{\mu_\phi}{kT} \qquad (2.10')$$

Denoting by Γ_ϕ the coefficient in the equation connecting the rate of change of N_ϕ and the quantity μ_ϕ, we have according to (2.10')

$$\Gamma_\phi = \frac{1}{kT} \iiint n_{10} n_{20} n_{30} (n_{40}+1)(n_{50}+1) \, d\omega \; \frac{dp_1 \, dp_2 \, dp_3}{(2\pi\hbar)^9} \qquad (2.12)$$

Without great loss of precision we may neglect the distribution functions n_{40} and n_{50} compared with unity in the integrands in (2.12). The integration over the phase volumes of the momenta of the three colliding phonons ($\underline{p_1}$, $\underline{p_2}$, and $\underline{p_3}$) can then be carried out, which allows $d\omega$ to be replaced by $\overline{\omega}$, reducing the number of integrations by one. We obtain in this way

$$\Gamma_\phi \approx \frac{1}{kT} \iiint n_{10} n_{20} n_{30} \; \overline{\omega} \; \frac{d\rho_1 \, d\rho_2 \, d\rho_3}{(2\pi h)^9} \qquad (2.12')$$

The integrand in (2.12') is proportional to p^{12} . After integrating (2.12') over the momenta of the colliding phonons we obtain for the temperature law for Γ_ϕ :

$$\Gamma_\phi = a T^{11} \qquad (2.13)$$

where a is a temperature-independent coefficient.

The expression (2.13) determines the temperature variation of just the quantity which enters significantly into our further calculations.

Emission and absorption of rotons.

A collision between two rotons may be associated with emission or absorption of a third roton. Accurate expressions for the probability of processes of this kind cannot be obtained since the interaction function between two rotons is not known.

The energy $\mathcal{E}$ of a roton of momentum $\underline{p}$ is known (5) to be equal to

$$\mathcal{E} = \Delta + (p-p_0)^2/2\mu$$

where Δ , p_0 and μ are parameters.

If two rotons with momenta $\underline{p_1}$ and $\underline{p_2}$ collide, a third roton can be emitted only if the condition

$$(1/2\mu) \left[(P_1-p_0)^2 + (p_2-p_0)^2 \right] \geqslant \Delta \qquad (2.14)$$

is complied with; this follows from the conservation law for the energy.

It follows from the form of the energy spectrum that most of the rotons have momenta of the magnitude p_0. Therefore condition (2.14) is very stringent. It may be

is extremely small.

Let N_p be the number of rotons per unit volume; if it is not equal to the equilibrium density then the rate of approach to equilibrium N_p, may in a similar way to that used previously be written as

$$\dot{N}_p = -\Gamma_p \mu_p \qquad (2,15)$$

where μ_p is the chemical potential of the roton gas.

If, as asserted, the probability for a five-roton-process is very small, then the quantity Γ_p must also be small. It is accordingly assumed that

$$\Gamma_p \ll \Gamma_\phi \qquad (2.16)$$

at all temperatures below the λ-point.

Transformation of phonons into rotons.

Transformations of rotons into phonons are forbidden on account of the conservation of momentum, in so far as the momentum of phonons (p) satisfy the condition

$$p \ll p_0$$

and simultaneously the momentum of the rotons is almost equal to p_0. This selection rule applies to the collision between two rotons if the angles between the momenta of the colliding rotons are small and hence their transformation into a scattered roton and phonon is impossible. If, however, the above angle is not too small the selection rule does not apply; collisions between two rotons are permitted in which they are transformed into a phonon and a roton and even into two phonons (if the angles between the momenta of the colliding rotons are near to π). For the reasons given a couple of paragraphs back, it is impossible to calculate the probabilities of these processes. It is possible to carry out estimates, just as before; the essential points of them are presented in the following.

Consider, for the sake of expediency, a process which is the inverse to the above collision between two rotons; let a phonon of momentum p_1 and a roton of momentum p_2 collide, and let this process result in two rotons of momenta p_3 and p_4 respectively. The energy condition demands that the energy of the phonon is not smaller than . The phonon must

accordingly have high energy. It is to be expected that phonons of energies as high as those considered differ from rotons only by the way in which the energy depends upon the momentum. Their interaction with rotons is, in particular, of the same kind as the interaction between rotons. Hence calculations can be carried out in full analogy to those appertaining to the scattering of a roton by a roton (3). Similarly as in this case the phonon-roton interaction energy V (the phonon having large energy, in any case greater than Δ) can be represented as being proportional to a delta-function of the distance between phonon and roton

$$V = V_0 \, \delta(\underline{r}_1 - r_2) \qquad\qquad (2,17)$$

(here $\underline{r}_1, \underline{r}_2$ are the position vectors of the phonon and roton respectively). It was stated previously (3) that an expression of this kind for the energy of interaction is chosen because it makes it possible to determine the temperature effect on the probability of a corresponding collision process. Just as before the rate of approach from non-equilibrium to equilibrium in a phonon-roton gas can be written in the form

$$\dot{N}_\rho = -\overline{\Gamma}_{\phi\rho}\left(\mu_\rho - \mu_\phi\right) , \quad \dot{N}_\phi = \Gamma_{\phi\rho}\left(\mu_\rho - \mu_\phi\right). \quad (2,18)$$

provided that the approach to equilibrium depends on process of transformation of rotons into phonons and vice-versa.

The coefficient $\Gamma_{\phi\rho}$ is determined by means of a complete calculation of the collision integral. Omitting intermediate steps in the calculation, we give the final result as obtained with the interaction function (2,17)

$$\Gamma_{\phi\rho} = \left|V_0\right|^2 \Delta^2 N_\rho^2 / \pi h^4 c^3 k \qquad (2,19)$$

The magnitude of the amplitude V_0 is not known for the process under discussion. For the purpose of making an estimate one can use the value for V_0 derived for the roton-roton interaction from the empirical viscosity of helium II (3). It may be noted that this estimate provides an upper limit for V_0 : for it is implied that the transformation of a phonon into a roton occurs in every process in which the phonon has sufficient energy and the angle between the momenta

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clearly this is a necessary but not a sufficient condition, for it allows also simple scattering of the phonon by the roton.

Inserting into the right hand side of (2,19) the values of all parameters and the assumed value for V_o (= 5 . 10^{-39} erg/cm^3) we find

$$\Gamma_{\phi\rho} \lesssim 3. \; 10^{50} \; e^{-2\Delta/T} \qquad (2,20)$$

A five - phonon process is obviously less probable than a four-phonon process. For the latter we made previously [3] estimates of sufficient accuracy, thereby providing an upper limit for Γ_ϕ. Comparison of these upper limits ($\Gamma_{\phi\rho}$ and Γ_ϕ) shows that they are of similar magnitude for wide range of temperature. Hence the two coefficients $\Gamma_{\phi\rho}$ and Γ_ϕ themselves are possibly of magnitudes similar to each other.

The numerical coefficient entering into the expression for $\Gamma_{\phi\rho}$ which is for the time being considered as unknown, is denoted by b, so that (2,19) reads

$$\Gamma_{\phi\rho} = b \; e^{-2\Delta/T} \qquad (2,21)$$

The quantity b and the numerical coefficient a of equation (2,13) are subsequently obtained from empirical data, the absorption coefficient for first sound in helium II. It will then be seen that $\Gamma_{\phi\rho}$ and Γ_ϕ have in fact the same order of magnitude.[+)]

3. Equations governing the propagation of sound in helium II.

The propagation of sound in helium II is determined by equations which may be written in the form of a linearized set of hydrodynamical equations of motion [5]:

[+)] In so far as it is due to phonons, viscosity is, as shown in [3], for temperatures between 1.5 and 0.7 deg.K dependent on the fundamental process of scattering of phonons by rotons. The probability of this process is known. Calculation of the collision integral for that process and the corresponding coefficient Γ and comparison with the upper limits of $\Gamma_{\phi\rho}$ and Γ_ϕ shows that $\Gamma_{\phi\rho} < \Gamma$ In this way it is justified to neglect the phonon-roton transformation in calculating the viscosity according to [3].

$$\partial\rho/\partial t + \text{div}\, \underline{j} = 0 \qquad (3,1)$$

$$\partial S/\partial t + S\, \text{div}\, \underline{v}_n = 0 \qquad (3,2)$$

$$\partial \underline{j}/\partial t + \nabla \rho = 0 \qquad (3,3)$$

$$\partial \underline{v}_s/\partial t + \nabla \overline{\Phi} = 0 \qquad (3,4)$$

Here ρ = density of helium, ρ_n and ρ_s = density and $\underline{v}_n$ and $\underline{v}_s$ = velocity of the normal and superfluid fraction respectively, S = entropy per unit volume, $\overline{\Phi}$ = thermodynamical potential per unit mass, $\underline{j}$ = $\rho_n \underline{v}_n + \rho_s \underline{v}_s$ = mass flow density.

In equilibrium, all thermodynamic functions appearing in equations $(3,1)-(3,4)$ depend on two variables only, which may be chosen to be ρ and S. If a sound wave of low frequency is propagated through helium II, equilibrium is re-established at such a high rate (as shown by a calculation) that the rate of approach to equilibrium follows instan- taneously after the change of state due to wave propagation. Hence the state is at every instant fully determined by the thermogynamical functions appropriate to equilibrium and dependent on ρ and S.

If, however, the frequency of the sound waves is so high that the approach to equilibrium cannot follow the change of state, then processes arise which have the effect of an approach to equilibrium and result in absorption of sound. In this range of frequencies, the number of phonons and rotons at any instant differs from the equilibrium value and the thermodynamic functions must now be considered as depending not only on ρ and S but also upon the number of phonons and rotons present (or rather upon the corresponding chemical potentials). The set of equations $(3,1)-(3,4)$ is accordingly to be considered as incomplete and should be supplemented by equations expressing the rate of change of the numbers of phonons and rotons.

Let N_p and N_ϕ be the number of phonons and rotons respectively per unit volume of helium II, μ_p and μ_ϕ their

chemical potentials. In equilibrium $\mu_p = \mu_\phi = 0$, and the

number of phonons and rotons denoted by N_{po} and $N_{\phi o}$ are functions of ρ and S only.

If equilibrium is perturbed, N_p and N_ϕ change in time approaching their equilibrium values N_{po} and $N_{\phi o}$. Equations determining the rate of approach to equilibrium can be obtained by expressing the time derivatives $\dot{N}_p$ and $\dot{N}_\phi$ as functions of the chemical potentials. Expanding this function in powers and concluding with the term linear in μ_ρ and μ_ϕ :

$$\dot{N}_p + \operatorname{div} N_p \underline{v}_n = -\gamma_{pp}\mu_p + \gamma_{p\phi}\,\mu_\phi \qquad (3,5)$$

$$\dot{N}_\phi + \operatorname{div} N_\phi \underline{v}_n = \gamma_{\phi p}\,\mu_p - \gamma_{\phi\phi}\,\mu_\phi \qquad (3,6)$$

where γ_{pp}, $\gamma_{\phi\phi}$, $\gamma_{\phi p}$, $\gamma_{p\phi}$ are the kinetic coefficients[+] which are symmetric with respect to the suffixes ϕ and p. In these equations the terms of the form $\operatorname{div} N\underline{v}_n$ account for the fact that phonons and rotons participate in the movement of the normal fraction of helium II with the velocity $\underline{v}_n$.

Neglecting second order effects, we substitute in the corresponding terms the equilibrium values of the N, so that (3,5) and (3,6) become

$$\dot{N}_p + N_{po}\operatorname{div}\underline{v}_n = -\gamma_{pp}\mu_p + \gamma_{p\phi}\,\mu_\phi \qquad (3,7)$$

$$\dot{N}_\phi + N_{\phi o}\operatorname{div}\underline{v}_n = \gamma_{\phi p}\,\mu_p - \gamma_{\phi\phi}\mu_\phi \qquad (3,8)$$

The set of equations (3,1)-(3,4), (3,7),(3,8) determines the propagation of sound in helium II taking due account of the change in time of the numbers of rotons and phonons. Elimination of two variables, $\underline{v}_n$ and $\underline{v}_s$ gives

$$\partial^2\rho/\partial t^2 = \Delta\rho \qquad (3,9)$$

$$\frac{\partial^2 S}{\partial t^2} = \frac{S}{\rho_n}\left(\Delta\rho - \rho_s\,\Delta\Phi\right) \qquad (3,10)$$

$$\frac{\partial N_\rho}{\partial t} - \frac{N_{\rho o}}{S}\frac{\partial S}{\partial t} = -\gamma_{\rho\rho}\,\mu_\rho + \gamma_{\rho\phi}\,\mu_\phi \qquad (3,11)$$

$$\frac{\partial N_\phi}{\partial t} - \frac{N_{\phi o}}{S}\frac{\partial S}{\partial t} = \gamma_{\phi\rho}\mu_\rho - \gamma_{\phi\phi}\mu_\phi \qquad (3,12)$$

[+] Cf. L. Landau and E. Lifshitz, Statistical Physics, § 41-42.

In equations (3,9)-(3,12) all thermodynamic functions can be
expressed as depending upon ρ, S, μ_p, μ_ϕ as independent
variables. Entropy and density can be represented as sums of
the equilibrium expression and a small increment due to the
sound waves; the increments are denoted by ρ', S'. A solu-
tion of this set of simultaneous equations is required which
has the form of a plane wave such that S', ρ', u_p and μ_ϕ
contain a factor exp $i\omega(t - x/u)$, where u is the velocity
of sound. The compatibility conditions for this set of
equations assumes the form of a determinant of 4th order being
equal to 0; solution of the latter equation provides 4 values
for the velocity of sound.[+)]

In order to avoid a determinant of the 4th order we
proceed in a slightly different way, by expressing all
functions in (3,9) and (3,10) in terms of S and ρ as
independent variables. Hereby the dependance of the function
upon the variables μ_p and μ_ϕ has to be taken into account at
all differentiations. The partial derivative of some thermo-
dynamic function f with respect to ρ is written

$$\left(\frac{\partial f}{\partial \rho}\right)_S = \left(\frac{\partial f}{\partial \rho}\right)_{S,\mu_p,\mu_\phi} + \left(\frac{\partial f}{\partial \mu_p}\right)\left(\frac{\partial \mu_p}{\partial \rho}\right)_S + \left(\frac{\partial f}{\partial \mu_\phi}\right)\left(\frac{\partial \mu_\phi}{\partial \rho}\right)_S , \quad (3,13)$$

Similarly

$$\left(\frac{\partial f}{\partial S}\right)_\rho = \left(\frac{\partial f}{\partial S}\right)_{\rho,\mu_p,\mu_\phi} + \left(\frac{\partial f}{\partial \mu_p}\right)\left(\frac{\partial \mu_p}{\partial S}\right)_\rho + \left(\frac{\partial f}{\partial \mu_\phi}\right)\left(\frac{\partial \mu_\phi}{\partial S}\right)_\rho . \quad (3,14)$$

The derivatives at constant chemical potentials (which
are equal to 0) are seen to be equal to the equilibrium value
of the derivative under consideration. In (3,13) and (3,14)

[+)]Attention is drawn to the fact that equations (3,9)-(3,12)
lead to a determinant of the 4th order, but that nevertheless
the equation determining the velocity u is a quadratic equa-
tion for u^2. Equations (3,11) and (3,12), which express the
variations of the number of quantized excitations do not
contain any differentiations of the variables with respect
to the coordinates. If a solution is sought which is propor-
tional to exp $i\omega(t - x/u)$, then the velocity u does not
enter into theseequations. The square of the velocity enters
only the first two lines of the determinant; it follows that
the characteristic equation is quadratic in u^2.

the derivatives of the chemical potentials μ_ρ and μ_ϕ
with respect to ρ and S are used. They are obtained by
means of equations (3,11) and (3,12) which establish rela-
tions between μ and μ on the one hand and the small varia-
tions of ρ and S in the plane wave on the other hand.

We express the small variations of ρ and S as before
including a factor exp $i\omega(t - x/u)$. The compatibility
condition for (3,9) and (3,10) is now obtained in the form
of a determinant of second order being equal to 0.

$$0 = \begin{vmatrix} u^2 - (\partial\rho/\partial\rho)_S & (\partial\rho/\partial S)_\rho \\ u^2 - \rho_S(\partial\Phi/\partial\rho)_S & \frac{\rho_n}{S}u^2 + \beta_S\left(\frac{\partial\Phi}{\partial S}\right)_\rho \end{vmatrix} \quad (3,15)$$

The quadratic equation which is obtained from (3,15) contains
terms of different magnitudes. All terms arising from the
differential coefficient $(\partial\rho/\partial S)_\rho$ turn out to be small.
This comes from the fact that the specific heats c_p and c_v of
helium II are almost equal to each other at all temperatures
and that the derivative $(\partial\rho/\partial S)_\rho$ is, according to a familiar
thermodynamic theorem, proportional to $(c_p - c_v)^{\frac{1}{2}}$. On account
of this, the roots of (3,15) can be written in a remarkably
concise form

$$u_1^2 = \left(\frac{\partial\rho}{\partial\rho}\right)_S + \left(\frac{\partial\rho}{\partial S}\right)_\rho \frac{S}{\rho} \quad (3,16)$$

$$u_2^2 = \frac{\beta_S S}{\rho_n}\left[\frac{1}{\rho}\left(\frac{\partial\rho}{\partial S}\right)_\rho - \left(\frac{\partial\Phi}{\partial S}\right)_\rho\right] \quad (3,17)$$

The chemical potentials μ_ρ and μ_ϕ can be expressed in
terms of the small increments of entropy and density (S' and
ρ') by means of equations (3,11) and (3,12), from which the
following two equations are derived

$$\left(i\omega\frac{\partial N_\rho}{\partial\mu_\rho} + \gamma_{\rho\rho}\right)\mu_\rho - \gamma_{\rho\phi}\mu_\phi = -i\omega\left(\frac{\partial N_\rho}{\partial\rho}\rho' + \frac{\partial N_\rho}{\partial S}S' - \frac{N_\rho}{S}S'\right),$$

$$-\gamma_{\phi\rho}\mu_\rho + \left(i\omega\frac{\partial N_\phi}{\partial\mu_\phi} + \gamma_{\phi\phi}\right)\mu_\phi = -i\omega\left(\frac{\partial N_\phi}{\partial\rho}\rho' + \frac{\partial N_\phi}{\partial S}S' - \frac{N_\phi}{T}S'\right)$$

It is readily found that

$$\mu_\rho = \frac{i\omega}{D(\omega)}\left\{\left(i\omega \frac{\partial N_\phi}{\partial \mu_\phi} + \gamma_{\phi\phi}\right)\left(\frac{N_\rho}{S}S' - \frac{\partial N_\rho}{\partial S}S' - \frac{\partial N_\rho}{\partial \rho}\rho'\right) + \gamma_{\rho\phi}\left(\frac{N_\phi}{S}S' - \frac{\partial N_\phi}{\partial S}S' - \frac{\partial N_\phi}{\partial \rho}\rho'\right)\right\} \tag{3,18}$$

$$\mu_\phi = \frac{i\omega}{D(\omega)}\left\{\left(i\omega \frac{\partial N_\rho}{\partial \mu_\rho} + \gamma_{\rho\rho}\right)\left(\frac{N_\phi}{S}S' - \frac{\partial N_\phi}{\partial S}S' - \frac{\partial N_\phi}{\partial \rho}\rho'\right) + \gamma_{\phi\rho}\left(\frac{N_\rho}{S}S' - \frac{\partial N_\rho}{\partial S}S' - \frac{\partial N_\rho}{\partial \rho}\rho'\right)\right\} \tag{3,19}$$

where
$$D(\omega) = \left(i\omega \frac{\partial N_\rho}{\partial \mu_\rho} + \gamma_{\rho\rho}\right)\left(i\omega \frac{\partial N_\phi}{\partial \mu_\phi} + \gamma_{\phi\phi}\right) - \gamma_{\rho\phi}\gamma_{\phi\rho}$$

For further calculation the thermodynamic identity

$$T\,dS + \Phi\,d\rho - N_\rho\,d\mu_\rho - N_\phi\,d\mu_\phi = d\Sigma \tag{3,20}$$

is required. The function Σ plays a similar part as the energy and, if the chemical potentials are constant, becomes equal to the energy per unit of volume of the liquid. A connection between the pressure p and the function is derived by remembering that the derivative of the total energy with respect to the volume at constant total <u>entropy and mass is</u> equal to · minus the pressure.

It then follows that[+)]

$$p = -\Sigma + ST + \Phi\rho \tag{3,21}$$

Here follow a number of thermodyanical relations which follow from (3,20), (3,21) and are required for calculating the derivatives in (3,16), (3,17):—

$$\left(\frac{\partial p}{\partial \mu_\rho}\right)_{\rho,S} = -\left(\frac{\partial \Sigma}{\partial \mu_\rho}\right)_{\rho,S} + S\left(\frac{\partial T}{\partial \mu_\rho}\right)_{\rho,S} + \rho\left(\frac{\partial \Phi}{\partial \mu_\rho}\right)_{\rho,S} = N_\rho - S\left(\frac{\partial N_\rho}{\partial S}\right)_{\rho,\mu} - \rho\left(\frac{\partial N_\rho}{\partial \rho}\right)_{S,\mu} \tag{30}$$

$$\left(\frac{\partial p}{\partial \mu_\phi}\right)_{\rho,S} = -\left(\frac{\partial \Sigma}{\partial \mu_\phi}\right)_{\rho,S} + S\left(\frac{\partial T}{\partial \mu_\phi}\right)_{\rho,S} + \rho\left(\frac{\partial \Phi}{\partial \mu_\phi}\right)_{\rho,S} = N_\phi - S\left(\frac{\partial N_\phi}{\partial S}\right)_{\rho,\mu} - \rho\left(\frac{\partial N_\phi}{\partial \rho}\right)_{S,\mu}$$

$$\left(\frac{\partial \Phi}{\partial \mu_\rho}\right)_{\rho,S} = -\left(\frac{\partial N_\rho}{\partial \rho}\right)_{S,\mu}, \qquad \left(\frac{\partial \Phi}{\partial \mu_\phi}\right)_{\rho,S} = -\left(\frac{\partial N_\phi}{\partial \rho}\right)_{S,\mu} \tag{3,23}$$

[+)] As a matter of principle, $p = -\frac{\partial(\Sigma V)}{\partial V} = -\Sigma - V\frac{\partial \Sigma}{\partial V} = -\Sigma - V\left(T\frac{\partial S}{\partial V} + \Phi\frac{\partial \rho}{\partial V}\right)$

As the mass ρV and the total entropy SV are constant:

$$\frac{\partial S}{\partial V} = -\frac{S}{V}, \quad \frac{\partial \rho}{\partial V} = -\frac{\rho}{V} \quad \left(=-\frac{S}{V} - \frac{\Phi\rho}{V}\right) = ST + \Phi\rho$$

The derivatives of the chemical potentials with respect to
ρ and S, on which the velocities of sound u_1 and u_2 depend
according to (3,13), (3,14), are obtained by means of
(3,18), (3,19):

$$\left(\frac{\partial \mu_\rho}{\partial \rho}\right)_S = \frac{i\omega}{D(\omega)}\left\{-\left(i\omega\frac{\partial N_\rho}{\partial \mu_\varphi}+\gamma_{\varphi\varphi}\right)\left(\frac{\partial N_\rho}{\partial \rho}\right)_S + \gamma_{\rho\varphi}\left(\frac{\partial N_\varphi}{\partial \rho}\right)_S\right\};$$

$$\left(\frac{\partial \mu_\varphi}{\partial \rho}\right)_S = \frac{i\omega}{D(\omega)}\left\{-\left(i\omega\frac{\partial N_\rho}{\partial \mu_\varphi}+\gamma_{\rho\rho}\right)\left(\frac{\partial N_\varphi}{\partial \rho}\right)_S + \gamma_{\varphi\rho}\left(\frac{\partial N_\rho}{\partial \rho}\right)_S\right\};$$

$$\left(\frac{\partial \mu_\rho}{\partial S}\right)_\rho = \frac{i\omega}{D(\omega)}\left\{\left(i\omega\frac{\partial N_\varphi}{\partial \mu_\varphi}+\gamma_{\varphi\varphi}\right)\left[\frac{N_\rho}{S}-\left(\frac{\partial N_\rho}{\partial S}\right)_\rho\right]-\gamma_{\rho\varphi}\left[\frac{N_\varphi}{S}-\left(\frac{\partial N_\varphi}{\partial S}\right)_\rho\right]\right\};$$

$$\left(\frac{\partial \mu_\varphi}{\partial S}\right)_\rho = \frac{i\omega}{D(\omega)}\left\{\left(i\omega\frac{\partial N_\rho}{\partial \mu_\rho}+\gamma_{\rho\rho}\right)\left[\frac{N_\varphi}{S}-\left(\frac{\partial N_\varphi}{\partial S}\right)_\rho\right]-\gamma_{\varphi\rho}\left[\frac{N_\rho}{S}-\left(\frac{\partial N_\rho}{\partial S}\right)_\rho\right]\right\};$$

(3,24)

Expressions (3,22)-(3,25) for the derivatives of the
thermodynamical quantities are readily used for obtaining
the velocities of sound u_1 and u_2. From (3,13), (3,14), (3,16) and (3,17),

$$u_1^2 = \left[\left(\frac{\partial \rho}{\partial \rho}\right)_S + \frac{S}{\rho}\left(\frac{\partial \rho}{\partial S}\right)_\rho\right]_{\mu_\rho=\mu_\varphi=0} + \frac{i\omega}{D(\omega)}\left[\left(i\omega\frac{\partial N_\varphi}{\partial \mu_\varphi}+\gamma_{\varphi\varphi}\right)\left(\frac{\partial \rho}{\partial \mu_\rho}\right)^2\right.$$
$$\left. + 2\gamma_{\rho\varphi}\frac{\partial \rho}{\partial \mu_\rho}\frac{\partial \rho}{\partial \mu_\varphi} + \left(i\omega\frac{\partial N_\rho}{\partial \mu_\rho}+\gamma_{\rho\rho}\right)\left(\frac{\partial \rho}{\partial \mu_\varphi}\right)^2\right]_{\rho,S}$$

$$u_2^2 = \frac{\rho_S}{\rho_n}S\left[\frac{1}{\rho}\left(\frac{\partial \rho}{\partial S}\right)_\rho - \left(\frac{\partial \Phi}{\partial S}\right)_\rho\right]_{\mu_\rho=\mu_\varphi=0} +$$

$$+ \frac{i\omega}{D(\omega)}\left[\left(i\omega\frac{\partial N_\varphi}{\partial \mu_\varphi}+\gamma_{\varphi\varphi}\right)\left(\frac{\partial \rho}{\partial \mu_\rho}-\rho\frac{\partial \Phi}{\partial \mu_\rho}\right)^2 + 2\gamma_{\rho\varphi}\left(\frac{\partial \rho}{\partial \mu_\rho}-\rho\frac{\partial \Phi}{\partial \mu_\rho}\right)\times\right.$$

$$\left.\times\left(\frac{\partial \rho}{\partial \mu_\varphi}-\rho\frac{\partial \Phi}{\partial \mu_\varphi}\right)+\left(i\omega\frac{\partial N_\rho}{\partial \mu_\rho}+\gamma_{\rho\rho}\right)\left(\frac{\partial \rho}{\partial \mu_\varphi}-\rho\frac{\partial \Phi}{\partial \mu_\varphi}\right)^2\right]_{\rho,S}$$

If the chemical potentials are set equal to 0, the above
expressions for the velocity of 1st and 2nd sound in helium
II apply to thermal equilibrium; this corresponds to extremely
low frequencies. These velocities are denoted by a suffix 0;
their values follow from (3,26), (3,27):

$$u_{1_0}^2 = \left[\left(\frac{\partial p}{\partial \rho}\right)_S + \frac{S}{\rho}\left(\frac{\partial p}{\partial S}\right)_\rho\right]_{\mu_\rho = \mu_\phi = 0} = \left(\frac{\partial p}{\partial \rho}\right)_\sigma \qquad (3.28)$$

$$u_{2_0}^2 = \frac{\rho_s}{\rho_n} S \left[\frac{1}{\rho}\left(\frac{\partial p}{\partial S}\right)_\rho - \left(\frac{\partial \Phi}{\partial S}\right)_\rho\right]_{\mu_\rho = \mu_\phi = 0} = \frac{\rho_s S^2}{\rho \rho_n}\left(\frac{\partial T}{\partial S}\right)_\rho \qquad (3.29)$$

Here $\sigma = S/\rho$. Expressions (3,28), (3,29) are equal to
expressions as obtained previously [5]. According to (3,26),
(3,27) the velocities u_1 and u_2 are complex. The wave vectors,
as defined according to $k = \omega/u$ are also complex. The
existence of an imaginary part of the wave vector is known to
be an expression of the fact that there is absorption of sound.
The real part of the wave vector determines the variation of
phase with the distance, whereas the imaginary part determines
the absorption coefficient.

If the frequencies are extremely high, the expressions
(3,26), (3,27) approach the limits:

$$u_{1_{00}}^2 = u_{1_0}^2 + \frac{1}{\rho}\left[\left(\frac{\partial p}{\partial \mu_\rho}\right)^2 \Big/ \left(\frac{\partial N_\rho}{\partial \mu_\rho}\right) + \left(\frac{\partial p}{\partial \mu_\phi}\right)^2 \Big/ \left(\frac{\partial N_\phi}{\partial \mu_\phi}\right)\right]_{\rho, S} \qquad (3.30)$$

$$u_{2_{00}}^2 = u_{2_0}^2 + \frac{\rho_s}{\rho \rho_n}\left[\left(\left(\frac{\partial p}{\partial \mu_\rho}\right) - \rho\frac{\partial \Phi}{\partial \mu_\rho}\right)^2 \Big/ \left(\frac{\partial N_\rho}{\partial \mu_\rho}\right) + \left(\frac{\partial p}{\partial \mu_\phi} - \rho\frac{\partial \Phi}{\partial \mu_\phi}\right)^2 \Big/ \left(\frac{\partial N_\phi}{\partial \mu_\phi}\right)\right]_{\rho, S} \qquad (3.31)$$

These velocities of sound apply to such high frequencies that
there is not time enough for processes in which the numbers of
phonons and rotons change; these numbers are then constant.

The kinetic coefficients γ which appear in the formulae
of this section are readily expressed in terms of those
quantities which are obtained in the preceding section, and
which determine the rate of inelastic scattering processes.

By comparison of equations (3,5) and (3,6) with equations (2,15) and (2,18), and considering the meaning of $\overline{\Gamma_\phi}$, we find

$$\gamma_{\rho\rho} = \Gamma_\rho + \Gamma_{\phi\rho} \quad, \quad \gamma_{\rho\phi} = \gamma_{\phi\rho} = \Gamma_{\phi\rho} \quad, \quad \gamma_{\phi\phi} = \overline{\Gamma_\phi} + \Gamma_{\phi\rho} \quad (3,3)$$

In the subsequent calculations it is convenient to employ instead of the coefficients γ and Γ the relaxation times defined according to

$$\frac{1}{\theta_\rho} = \Gamma_\rho \left(\frac{\partial \mu_\rho}{\partial N_\rho}\right)_{\rho,S} \quad, \quad \frac{1}{\theta_{\rho\phi}} = \left(\frac{\partial \mu_\rho}{\partial N_\rho} + \frac{\partial \mu_\phi}{\partial N_\phi}\right)_{\rho,S} \quad, \quad \frac{1}{\theta_\phi} = \Gamma_\phi \left(\frac{\partial \mu_\phi}{\partial N_\phi}\right)_{\rho,S} \quad (3.)$$

By means of the definitions (3,33), the expressions on the right-hand side of (3,26) and (3,27) are converted to

$$u_1^2 = u_{10}^2 + \frac{i\omega}{\rho} \frac{\left[\left(i\omega + \frac{1}{\theta_\rho}\right)\left(\frac{\partial \rho}{\partial \mu_\rho}\right)^2 \left(\frac{\partial N_\rho}{\partial \mu_\rho}\right) + \frac{1}{\theta_{\rho\phi}}\left(\frac{\partial \rho}{\partial \mu_\rho} + \frac{\partial \rho}{\partial \mu_\phi}\right)^2\left(\frac{\partial N_\rho}{\partial \mu_\rho} + \frac{\partial N_\phi}{\partial \mu_\phi}\right) + \left(i\omega + \frac{1}{\theta_\rho}\right)\left(\frac{\partial \rho}{\partial \mu_\phi}\right)^2\left(\frac{\partial N_\phi}{\partial \mu_\phi}\right)\right]_{\rho,S}}{\left(i\omega + \frac{1}{\theta_\rho}\right)\left(i\omega + \frac{1}{\theta_\phi}\right) + \frac{1}{\theta_{\rho\phi}}\left[i\omega + \left(\frac{1}{\theta_\rho}\frac{\partial N_\rho}{\partial \mu_\rho} + \frac{1}{\theta_\phi}\frac{\partial N_\phi}{\partial \mu_\phi}\right)\left(\frac{\partial N_\rho}{\partial \mu_\rho} + \frac{\partial N_\phi}{\partial \mu_\phi}\right)\right]_{\rho,S}} \quad (3.4)$$

$$u_2^2 = u_{20}^2 + \frac{i\omega}{\rho} \frac{\left[\left(i\omega + \frac{1}{\theta_\phi}\right)\left(\frac{\partial \rho}{\partial \mu_\rho} - \rho\frac{\partial \phi}{\partial \mu_\rho}\right)^2\left(\frac{\partial N_\rho}{\partial \mu_\rho}\right) + \frac{1}{\theta_{\rho\phi}}\left(\frac{\partial \rho}{\partial \mu_\rho} + \frac{\partial \rho}{\partial \mu_\phi} - \rho\frac{\partial \phi}{\partial \mu_\rho} - \rho\frac{\partial \phi}{\partial \mu_\phi}\right)^2\left(\frac{\partial N_\rho}{\partial \mu_\rho} + \frac{\partial N_\phi}{\partial \mu_\phi}\right)\right]_{\rho,S}}{\left(i\omega + \frac{1}{\theta_\rho}\right)\left(i\omega + \frac{1}{\theta_\phi}\right) + \frac{1}{\theta_{\rho\phi}}\left[i\omega + \left(\frac{1}{\theta_\rho}\frac{\partial N_\rho}{\partial \mu_\rho} + \frac{1}{\theta_\phi}\frac{\partial N_\phi}{\partial \mu_\phi}\right)\left(\frac{\partial N_\rho}{\partial \mu_\rho} + \frac{\partial N_\phi}{\partial \mu_\phi}\right)\right]_{\rho,S}} +$$

$$+ \frac{i\omega}{\rho} \frac{\left[\left(i\omega + \frac{1}{\theta_\rho}\right)\left(\frac{\partial \rho}{\partial \mu_\phi} - \rho\frac{\partial \phi}{\partial \mu_\phi}\right)^2\left(\frac{\partial N_\phi}{\partial \mu_\phi}\right)\right]_{\rho,S}}{\left(i\omega + \frac{1}{\theta_\rho}\right)\left(i\omega + \frac{1}{\theta_\phi}\right) + \frac{1}{\theta_{\rho\phi}}\left[i\omega + \left(\frac{1}{\theta_\rho}\frac{\partial N_\rho}{\partial \mu_\rho} + \frac{1}{\theta_\phi}\frac{\partial N_\phi}{\partial \mu_\phi}\right)\left(\frac{\partial N_\rho}{\partial \mu_\rho} + \frac{\partial N_\phi}{\partial \mu_\phi}\right)\right]_{\rho,S}} \quad (3.35)$$

By these equations the velocities of 1st and 2nd sound are determined as a function of the frequency. In this way the coefficient of absorption of 1st and 2nd sound in helium II can be calculated.

It has, however, to be pointed out that equations (3,34) and (3,35) are not sufficient for determining the dispersion of sound in helium II. The problem of dispersion is complicated by the rapid increase of the mean free path of the quantized excitations at decreasing temperature.

Comparing the time constant θ_η , by which the first viscosity is specified $(^3)$, with the time constants $\theta_{\rho\varphi}$ and θ_φ, it is found that these time constants are at all temperatures of equal order of magnitude. The range of idspersion for sound coincides with the range of dispersion of the first viscosity. If this first fact is taken in account, it is necessary to apply some specialized arguments which we are going to postpone for a future paper. Equations (3,34) and (3,35) are accordingly applied to the calculation of the absorption coefficient only; they are considered to be valid for 1st and 2nd sound at low frequencies.$^{+)}$

The range of frequencies where the formulae for $u_{1\infty}$ and $u_{2\infty}$ are valid is presumably out of reach. Even at frequencies lower than these any propagation of sound waves is precluded by the phenomenon of the mean free path of phonons and rotons approaching the same magnitude as the wave length of the sound wave. This is another point to be studied more closely in some other paper.

$^{+)}$ It may be noted that in first approximation with respect to ω there is no dispersion.

4. Absorption of first sound in helium II.

Consider the range of frequencies complying with the conditions

$$\omega \theta_\rho \ll 1 \quad , \quad \omega \vartheta_{\rho\varphi} \ll 1 \qquad (4.1)$$

It will be shown subsequently that the derivatives $\left(\partial N_\rho / \partial \mu_\rho\right)_{\rho,S}$ and $\left(\partial N_\varphi / \partial \mu_{\rho\varphi}\right)_{\rho,S}$ are of similar magnitude. What has been said in section 2 with respect to the relative magnitudes of the coefficient Γ_ρ accordingly remains valid for the reciprocal time constant θ_ρ , which is determined according to (3,33). For this reason we are entitled to neglect in (3,34) and (3,35) the terms that are due to the five-roton process. Expression (3,35) then assumes at low frequencies the simplified form

$$u_1^2 = u_{10}^2 + \frac{i\omega}{\rho}\left[\theta_\rho\left(\frac{\partial\rho}{\partial\mu_\rho}+\frac{\partial\rho}{\partial\mu_\varphi}\right)^2\frac{\partial\mu_\varphi}{\partial N_\varphi} + \theta_{\rho\varphi}\left(\frac{\partial\rho}{\partial\mu_\varphi}\right)^2\left(\frac{\partial\mu_\rho}{\partial N_\rho}+\frac{\partial\mu_\varphi}{\partial N_\varphi}\right)\right]_{\rho,S} \quad (4.2)$$

The imaginary part of the wave-vector can be obtained by means of the last equation; the coefficient of absorption for first sound[+] denoted by $\widetilde{\alpha}_1$, is given by

$$\widetilde{\alpha}_1 = Im\left(\frac{\omega}{u_1}\right) = \frac{\omega^2}{2\rho c^3}\left[\theta_\rho\left(\frac{\partial\rho}{\partial\mu_\rho}+\frac{\partial\rho}{\partial\mu_\varphi}\right)^2\frac{\partial\mu_\varphi}{\partial N_\varphi} + \theta_{\rho\varphi}\left(\frac{\partial\rho}{\partial\mu_\varphi}\right)^2\left(\frac{\partial\mu_\rho}{\partial N_\rho}+\frac{\partial\mu_\varphi}{\partial N_\varphi}\right)\right]_{\rho,S} \quad (4.3)$$

In order to compare (4,3) with the experimental data given in ([1]), it is necessary to evaluate the differential coefficients entering into that expression. It is convenient to introduce ρ and T as independent variables. From (3,22) we derive

$$\left(\frac{\partial\rho}{\partial\mu_\rho}\right)_{\rho,S} = \left[N_\rho - \rho\left(\frac{\partial N_\rho}{\partial\rho}\right)_T\right] - \left(\frac{\partial N_\rho}{\partial T}\right)_\rho\left(\frac{\partial T}{\partial S}\right)_\rho\left[S - \rho\left(\frac{\partial S}{\partial\rho}\right)_T\right] \qquad (4.4)$$

$$\left(\frac{\partial\rho}{\partial\mu_\varphi}\right)_{\rho,S} = \left[N_\rho - \rho\left(\frac{\partial N_\varphi}{\partial\rho}\right)_T\right] - \left(\frac{\partial N_\varphi}{\partial T}\right)_\rho\left(\frac{\partial T}{\partial S}\right)_\rho\left[S - \rho\left(\frac{\partial S}{\partial\rho}\right)_T\right] \qquad (4.5)$$

[+] The symbol u_{10} is in the following replaced by the conventional c.

Calculations are fairly simple (cf. appendix).

Here are the results for $\left(\partial \mu_\rho / \partial N_\rho\right)_{\rho,S}$ and $\left(\partial \mu_\varphi / \partial N_\varphi\right)_{\rho,S}$.

$$\left(\frac{\partial \mu_\rho}{\partial N_\rho}\right)_{\rho,S} = \frac{kT}{N_\rho}\left[\left(\frac{\Delta^2}{T^2} + \frac{\Delta}{T} + \frac{3}{4}\right)\frac{N_\rho}{N_\varphi} + \frac{\pi^4}{9}\right] \Big/ \left[\frac{1}{2}\frac{N_\rho}{N_\varphi} + \frac{\pi^4}{9}\right], \quad (4\cdot6)$$

$$\left(\frac{\partial \mu_\varphi}{\partial N_\varphi}\right)_{\rho,S} \approx \frac{36}{5\pi^2}\frac{kT}{N_\varphi}\left[\frac{\Delta^2}{T^2} + \frac{\Delta}{T} + \frac{3}{4}\right)\frac{N_\rho}{N_\varphi} + \frac{\pi^4}{9}\right] \Big/ \left[\left(\frac{\Delta^2}{T^2} + \frac{\Delta}{T} + \frac{3}{4}\right)\frac{N_\rho}{N_\varphi} + \frac{\pi^4}{9}\frac{324}{5\pi^2}\right]. \quad (4\cdot7)$$

It should be noted that the number of rotons between 1.5 and 2 deg is markedly higher than the number of phonons ($N_\rho > N_\varphi$) but that the derivative $\left(\partial \mu_\varphi / \partial N_\varphi\right)_{\rho,S}$ is smaller than $\left(\partial \mu_\rho / \partial N_\rho\right)_{\rho,S}$; this is due to the factor Δ^2/T^2 in the denominator of (4,7).

We require expressions for the entropy S and for the number of phonons and rotons per volume unit in equilibrium

$$S = kN_\rho\left(\frac{\Delta}{T} + \frac{3}{2}\right) + kN_\varphi\frac{\pi^4}{27}, \quad (4\cdot8)$$

$$N_\rho = 2\,(2\pi)^{-3/2}\,p_0^2\,(\mu kT)^{1/2}\,\hbar^{-3}\,e^{-\Delta/T} \quad (4\cdot9)$$

$$N_\varphi = \frac{6}{5\pi^2}\,\left(\frac{kT}{c\hbar}\right)^3 \quad (4\cdot10)$$

Omitting intermediate steps in the calculation, we present the results:

$$\left(\frac{\partial \rho}{\partial \mu_\rho}\right)_{\rho,S} = \frac{N_\rho\left\{-\frac{\Delta}{T}\left(1 - \frac{\rho}{N_\rho}\frac{\partial N_\rho}{\partial \rho}\right)\frac{N_\rho}{N_\varphi} + \frac{\pi^4}{27}\left[3\left(1 - \frac{\rho}{N_\varphi}\frac{\partial N_\varphi}{\partial \rho}\right) - \left(\frac{1}{2} + \frac{\Delta}{T}\right)\left(1 - \frac{\rho}{N_\varphi}\frac{\partial N_\varphi}{\partial \rho}\right)\right]\right\}}{\left(\frac{\Delta^2}{T^2} + \frac{\Delta}{T} + \frac{3}{4}\right)\frac{N_\rho}{N_\varphi} + \frac{\pi^4}{9}} \quad (4\cdot11)$$

$$\left(\frac{\partial \rho}{\partial \mu_\varphi}\right)_{\rho,S} = \frac{N_\rho\left[\left(\frac{\Delta^2}{T^2} + \frac{\Delta}{T} + \frac{3}{4}\right)\left(1 - \frac{\rho}{N_\varphi}\frac{\partial N_\varphi}{\partial \rho}\right) - 3\left(\frac{\Delta}{T} + \frac{3}{2}\right)\left(1 - \frac{\rho}{N_\rho}\frac{\partial N_\rho}{\partial \rho}\right)\right]}{\left(\frac{\Delta^2}{T^2} + \frac{\Delta}{T} + \frac{3}{4}\right)\frac{N_\rho}{N_\varphi} + \frac{\pi^4}{9}} \quad (4\cdot12)$$

The data of Peshkov & Zinovieva [6] concerning the velocity of second sound under pressure can be used for calculating those derivatives of parameters which determine the value of $\partial N_\rho / \partial \rho$. The calculations have been given in a previous paper [3]. Assuming that u is virtually independent of ρ, the result is:

$$\frac{\rho}{\rho_0} \frac{\partial \rho_0}{\partial \rho} \approx 1 \quad , \quad \frac{\partial \Delta}{\partial \rho} \approx 0$$

On account of this and of equation (4,9), it is found that

$$\left(1 - \frac{\rho}{N_\rho} \frac{\partial N_\rho}{\partial \rho}\right) \approx -1 \qquad (4.13)$$

The differential coefficient of the velocity of first sound c with respect to the density ρ can be derived from Keesom's density - pressure curve for helium II, which yields $(\rho/c)(\partial c/\partial \rho) \approx 1.8$, and consequently

$$\left(1 - \frac{\rho}{N_\phi} \frac{\partial N_\phi}{\partial \rho}\right) = \left(1 + \frac{3\rho}{c} \frac{\partial c}{\partial \rho}\right) \approx 6.4 \qquad (4.14)$$

Using these values we obtain finally:

$$\left(\frac{\partial \rho}{\partial \mu_\rho}\right)_{P,S} = N_\rho \left[\frac{\Delta}{T}\left(\frac{N_\rho}{N_\phi} - 23.2\right) - 22.5\right] \Big/ \left[\left(\frac{\Delta^2}{T^2} + \frac{\Delta}{T} + \frac{3}{4}\right)\frac{N_\rho}{N_\phi} + \frac{\pi^4}{9}\right] \qquad (4.15)$$

$$\left(\frac{\partial \rho}{\partial \mu_\phi}\right)_{P,S} = N_\rho \left[6.4 \frac{\Delta^2}{T^2} + 9.4 \frac{\Delta}{T} + 9.3\right] \Big/ \left[\left(\frac{\Delta^2}{T^2} + \frac{\Delta}{T} + \frac{3}{4}\right)\frac{N_\rho}{N_\phi} + \frac{\pi^4}{9}\right] \qquad (4.16)$$

It is necessary to remember that equation (4,3) provides only that contribution to the absorption coefficient which is due to relaxation in connection with the creation of phonons and rotons. The complete absorption coefficient consists of two terms, of which one is given by (4,3) and the other is due to the effect of first viscosity and is equal to $\alpha = 2\omega^2 \eta / 3\rho c^2$

Pellam & Squire [1] made measurements of the absorption coefficient of first sound at a frequency of $1.5 \cdot 10^7 \text{ sec}^{-1}$ and at temperatures between 1.57 and 2.0 deg.K. Their data are, in spite of their small number, sufficient for calculating the coefficients a and b with sufficient accuracy. Introducing

these parameters into the formulae (2,13), (2,19), Γ_ϕ and $\Gamma_{\phi\rho}$ are obtained.

$$\Gamma_\phi \approx 6.10^{42}\, T^{11}, \quad \Gamma_{\phi\rho} \approx 1.2 \cdot 10^{50}\, e^{-2\Delta/T} \quad (4.18)$$

This expression for $\Gamma_{\phi\rho}$ is of course smaller than the upper limit given by (2,20). By means of (4,18) and the derivatives of the chemical potentials (4,6), (4,7) the relaxation times can be derived as function of the temperature.

In figure 1 the reciprocal relaxation times θ_ϕ and $\theta_{\phi\rho}$ are plotted against the temperature. This graph shows that in the experiments of Pellam and Squire [1] condition (4,1) was complied with at all temperatures. At the frequency applied by these

Fig 1
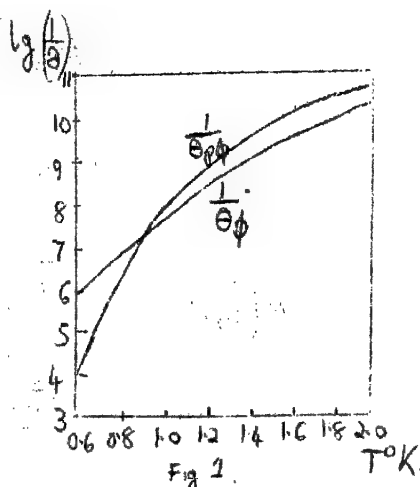
authors the greatest absorption should have been observed in the neighbourhood of 1°K. It is now easy to understand why in the experiments of Peshkov [4], which were made at frequencies between 10 and 10^4 sec^{-1}, no dispersion of second sound could be observed. In these experiments the temperature was appreciably higher (1.63 deg) so that the relaxation times θ_ϕ and $\theta_{\phi\rho}$ are as low as 10^{-10} sec. Hence $\omega\theta_\phi$ and $\omega\theta_{\phi\rho}$ were so small that there was not only no dispersion, but hardly any absorption of sound in helium II.

We add a table of the difference $u_{1\infty} - u_{10}$ as calculated from (3,30).

T (deg K)	0.6	0.7	0.8	0.9	1.0	1.1	1.2	
$u_{1\infty} - u_{10}$	0.175	1.45	5.80	13.5	23.0	37.5	51.0	
T	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
$u_{1\infty} - u_{10}$	73.5	90.5	115	140	165	195	240	275

The difference $u_{1\infty} - u_{10}$ is seen to be small[+]).

+) It vanishes when absolute zero is approached.

5. Absorption of second sound in helium II.

Using the results obtained in section 4, predictions concerning the effect of temperature on the absorption coefficient of second sound in helium II can be made. If the frequencies are small, equation (3,35) may be written, on account of condition (4,1)

$$u_2^2 = u_{20}^2 + \frac{i\omega\rho_s}{\rho_n\rho}\left[\theta_\phi\left(\frac{\partial\rho}{\partial\mu_\rho} + \frac{\partial\rho}{\partial\mu_\phi} - \rho\frac{\partial\Phi}{\partial\mu_\rho} - \rho\frac{\partial\Phi}{\partial\mu_\phi}\right)^2\frac{\partial\mu_\phi}{\partial N_\phi} + \right.$$
$$\left. + \theta_{\rho\phi}\left(\frac{\partial\rho}{\partial\mu_\rho} - \rho\frac{\partial\Phi}{\partial\mu_\rho}\right)^2\left(\frac{\partial\mu_\rho}{\partial N_\rho} + \frac{\partial\mu_\phi}{\partial N_\phi}\right)\right]_{\rho,S} \quad (5,1)$$

Hence the absorption coefficient for second sound is readily derived as

$$\alpha_2 = Im\left(\frac{\omega}{u_2}\right) = \frac{\omega^2\rho_s}{2\rho_n\rho u_{20}^3}\left[\theta_\phi\left(\frac{\partial\rho}{\partial\mu_\rho} + \frac{\partial\rho}{\partial\mu_\phi} - \rho\frac{\partial\Phi}{\partial\mu_\rho} - \rho\frac{\partial\Phi}{\partial\mu_\phi}\right)^2\frac{\partial\mu_\phi}{\partial N_\phi} + \right.$$
$$\left. + \theta_{\rho\phi}\left(\frac{\partial\rho}{\partial\mu_\rho} - \rho\frac{\partial\Phi}{\partial\mu_\rho}\right)^2\left(\frac{\partial\mu_\rho}{\partial N_\rho} + \frac{\partial\mu_\phi}{\partial N_\phi}\right)\right]_{\rho,S} \quad (5,2)$$

According to (3,22) and (3,23):

$$\left(\frac{\partial\rho}{\partial\mu_\phi} - \rho\frac{\partial\Phi}{\partial\mu_\phi}\right)_{\rho,S} = N_\rho - S\left(\frac{\partial N_\rho}{\partial S}\right)_\rho$$
$$\left(\frac{\partial\rho}{\partial\mu_\phi} - \rho\frac{\partial\Phi}{\partial\mu_\phi}\right)_{\rho,S} = N_\phi - S\left(\frac{\partial N_\phi}{\partial S}\right)_\rho \quad (5,3)$$

Evaluation of these expressions gives by means of relatively simple calculations:

$$\left(\frac{\partial\rho}{\partial\mu_\rho} - \rho\frac{\partial\Phi}{\partial\mu_\rho}\right)_{\rho,S} = -N_\rho\left[\frac{\Delta}{T}\left(\frac{N_\rho}{N_\phi} + \frac{\pi^4}{27}\right) - \frac{5\pi^4}{54}\right]\bigg/\left[\left(\frac{\Delta^2}{T^2} + \frac{\Delta}{T} + \frac{3}{4}\right)\frac{N_\rho}{N_\phi} + \frac{\pi^4}{9}\right] \quad (5,4)$$

$$\left(\frac{\partial\rho}{\partial\mu_\phi} - \rho\frac{\partial\Phi}{\partial\mu_\phi}\right)_{\rho,S} = N_\rho\left[\frac{\Delta^2}{T^2} - 2\frac{\Delta}{T} - \frac{15}{4}\right]\bigg/\left[\frac{\Delta^2}{T^2} + \frac{\Delta}{T} + \frac{3}{4}\right)\frac{N_\rho}{N_\phi} + \frac{\pi^4}{9}\right]$$

The densities ρ_s and ρ_n, which are required for evaluating (5,2), are according to ([5])

$$\rho_s = \rho - \rho_n \quad , \quad \rho_n = \left[\frac{N_\rho\rho_0^2}{3kT} + \frac{\pi^4}{27}\frac{N_\phi}{c^2}\frac{kT}{c^2}\right] \quad (5,5)$$

The right hand side of (5,2) expresses the rather involved relation between the temperature and the absorption coefficient. It must be noted that the absorption

to which it is due to the processes of creation of phonons and rotons. In the following section it will be shown that absorption of second sound owing to ordinary viscosity is of similar magnitude as that calculated in this section.

We add a table giving the velocity $u_{2\infty}$ at various temperatures[+].

T (deg.)	0.6	0.7	0.8	0.9	1.0	1.1	1.2
u_{20} (m/sec)	101	59.5	31.8	24.9	20.6	19.1	18.7
$u_{2\infty}$ (m/sec)	103	65.0	37.8	29.6	25.5	23.7	22.7

T	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
u_{20}	19.0	19.5	20.1	20.4	20.5	20.3	19.6	18.3
$u_{2\infty}$	22.9	23.1	23.9	24.1	24.5	24.3	23.8	20.8

6. Second viscosity

A classical liquid has two coefficients of viscosity such that the dissipation function appropriate to viscous flow has the form

$$ f = \frac{1}{2} \int \left\{ \frac{1}{2}\eta \left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} - \delta_{ik}\frac{\partial v_l}{\partial x_l} \right)^2 + \zeta \left(\operatorname{div} \underline{v} \right)^2 \right\} dV \quad (61) $$

where dV is a volume element in the liquid. Owing to the effect of dissipation there are 2 supplementary terms in the equations of motion, which have to be written:

$$ \rho \left[\frac{\partial \underline{v}}{\partial t} + (\underline{v}\nabla)\underline{v} \right] + \nabla_\rho = \eta \Delta \underline{v} + \left(\zeta + \frac{\eta}{3} \right) \nabla \operatorname{div} \underline{v} \quad (62) $$

Usually the first coefficient of viscosity η and the second coefficient ζ are of equal magnitude.

Matters are different with respect to helium II. Its hydrodynamics has to consider two velocities, that appropriate to normal movement $\underline{v}_n$ and the velocity of superfluid movement $\underline{v}_s$. This opens an unusual perspective. Whereas the first viscosity affects only the normal movement this is

not at all obvious with regard to the second coefficient.
In distinction from classical hydrodynamics, the hydrodynamics
of helium II involves two equations of motion, one referring
to the entire liquid

$$\frac{\partial j}{\partial t} + \nabla p = 0 \qquad (63)$$

one for the superfluid fraction

$$\frac{\partial v_s}{\partial t} + \nabla \Phi = 0 \qquad (64)$$

(where Φ is the thermodynamic potential for the unit of mass
of the liquid).

Consider now the equations of motion if second viscosity
is explicitly accounted for. In introducing the terms
depending on the second viscosity into (6,3), (6,4) we shall
for sake of simplicity omit at first the terms depending on
the first viscosity.

It is very obvious that the terms $\nabla \operatorname{div} \underline{v}$ require the
introduction of four coefficients of second viscosity.
Following the classical example (6,2), the right hand side
of (6,3) and (6,4) should involve two terms, $\nabla \operatorname{div} \underline{v}_n$ and
$\nabla \operatorname{div} \underline{v}_s$. Hence

$$\frac{\partial j}{\partial t} + \nabla p = \zeta_n \nabla \operatorname{div} \underline{v}_n + \zeta_s \nabla \operatorname{div} \underline{v}_s \qquad (65)$$

$$\rho_s \frac{\partial v_s}{\partial t} + \rho_s \nabla \Phi = \zeta_n^s \nabla \operatorname{div} \underline{v}_n + \zeta_s^s \nabla \operatorname{div} \underline{v}_s \qquad (66)$$

The unusual features in the hydrodynamics of helium
require accordingly the introduction of 4 "second" coefficients
of viscosity. These are, however, not independent. There
is a relation between them due to the fact that the part of
the dissipation function depending upon the second viscosity
should be quadratic function of the two variables $\operatorname{div} \underline{v}_n$ and
$\operatorname{div} \underline{v}_s$. It is therefore to be written in the form

$$f = \frac{1}{2} \int \left\{ A_n \left(\operatorname{div} \underline{v}_n\right)^2 + 2 A_{ns} \operatorname{div} \underline{v}_n \cdot \operatorname{div} \underline{v}_s + A_s \left(\operatorname{div} \underline{v}_s\right)^2 \right\} dV \qquad (6.7)$$

where A_n, A_{ns}, A_s are three coefficients which can be expressed in terms of the four viscosity coefficients.

The density of the negative derivative of f with respect to $\underline{v}_n$ and $\underline{v}_s$ should give the right hand side of the equations of motion for the normal and the superfluid fraction respectively.

(6.6) is the equation of motion for the superfluid fraction. The equation of motion for the normal fraction is obtained by subtracting (6,6) from (6,5).[+)

$$\rho_n \frac{\partial v_n}{\partial t} + \left(\nabla_\rho - \rho_s \nabla \Phi\right) = \left(\xi_n - \xi_n^s\right) \nabla \operatorname{div} \underline{v}_n + \left(\xi_s - \xi_s^s\right) \nabla \operatorname{div} \underline{v}_s \quad (6.8)$$

The derivatives $\partial f / \partial \underline{v}_n$ and $\partial f / \partial \underline{v}_s$ are readily calculated. For this purpose the variation of f in terms of variations of $\underline{v}_n$ and $\underline{v}_s$ is derived:

$$\delta f = \int \left\{ A_n \operatorname{div} \underline{v}_n \operatorname{div} \delta \underline{v}_n + A_{ns} \left(\operatorname{div} \underline{v}_n \operatorname{div} \delta \underline{v}_s + \operatorname{div} \underline{v}_s \operatorname{div} \delta \underline{v}_n \right) + \right.$$
$$\left. + A_s \operatorname{div} \underline{v}_s \operatorname{div} \delta \underline{v}_n \right\} dV$$

This expression is re-arranged:

$$\delta f = \int \left\{ A_n \left[\operatorname{div} \left(\delta v_n \operatorname{div} v_n \right) - \delta v_n \nabla \operatorname{div} v_n \right] + A_{ns} \left[\operatorname{div} \left(\delta v_s \operatorname{div} v_n \right) + \right. \right.$$
$$+ \operatorname{div} \left(\delta v_n \operatorname{div} v_s \right) - \delta v_s \nabla \operatorname{div} v_n - \delta v_n \nabla \operatorname{div} v_s \right] +$$
$$\left. + A_s \left[\operatorname{div} \left(\delta v_s \operatorname{div} v_s \right) - \delta v_s \nabla \operatorname{div} v_s \right] \right\} dV$$

All integrals which, according to the theorems of Gauss and Ostrogradsky, can be transformed into surface integrals, are considered to vanish. In this way we find

$$\delta f = -\int \left\{ A_n \delta v_n \nabla \operatorname{div} v_n + A_{ns} \left[\delta v_s \nabla \operatorname{div} v_n + \delta v_n \nabla \operatorname{div} v_s \right] + A_s \delta v_s \nabla \operatorname{div} v_s \right\} dV$$

Hence

$$\partial f / \partial v_n = -\int \left(A_n \nabla \operatorname{div} v_n + A_{ns} \nabla \operatorname{div} v_s \right) dV,$$
$$\left. \right\} \quad (6.9)$$
$$\partial f / \partial v_s = -\int \left(A_{ns} \nabla \operatorname{div} v_n + A_s \nabla \operatorname{div} v_s \right) dV$$

+) Terms of the form $\underline{v}_n \partial \rho_n / \partial t$ and $\underline{v}_s \partial \rho_s / \partial t$ are neglected in deriving (6,8).

The right hand sides of the equations of motion for the normal and the superfluid fraction are

$$A_{ns} \nabla \operatorname{div} v_n + A_s \nabla \operatorname{div} v_s, \tag{6.6'}$$

$$A_n \nabla \operatorname{div} v_n + A_{ns} \nabla \operatorname{div} v_s. \tag{6.8'}$$

Comparison of (6,6) and (6,8) with (6,6') and (6,8') respectively gives finally the relations between the coefficients A_n, A_{ns}, A_s, and the coefficients of second viscosity

$$A_{ns} = \zeta_n^s = \zeta_s - \zeta_s^s, \quad A_n = \zeta_n - \zeta_n^s, \quad A_s = \zeta_s^s \tag{6.12}$$

The first of these three equations yields the required relation between the coefficients of second viscosity.

We are now going to explain the relations between the thermodynamic quantities and the second viscosities. Since the second viscosity of helium II is due to processes involving changes in the total number of phonons and rotons, it can be concluded that the terms containing div $\underline{v}_n$ and div $\underline{v}_s$ in (6,5) and (6,6) are dependant on the relation of the pressure p and the potential ϕ, and the chemical potentials μ_ρ and μ_ϕ.

Therefore the right hand sides of (6,5) and (6,6) are equal to

$$-\left(\left(\frac{\partial \rho}{\partial \mu_\rho}\right) \nabla \mu_\rho + \left(\frac{\partial \rho}{\partial \mu_\phi}\right) \nabla \mu_\phi\right) \quad \text{and} \quad -\left(\left(\frac{\partial \phi}{\partial \mu_\rho}\right) \nabla \mu_\rho + \left(\frac{\partial \phi}{\partial \mu_\phi}\right) \nabla \mu_\phi\right) \text{respectively.}$$

By means of the expressions (3,18), (3,19) for μ_ρ and μ_ϕ, considering the inequality (4,5), it is found that

$$\mu_\rho = i\omega\left\{\frac{1}{I_\phi}\left(\frac{N}{S}S' - \frac{\partial N}{\partial S}S' - \frac{\partial N}{\partial \rho}\rho'\right) + \frac{1}{I_{\rho}}\left(\frac{N_\rho}{S}S' - \frac{\partial N_\rho}{\partial S}S' - \frac{\partial N_\rho}{\partial \rho}\rho'\right)\right\}$$

$$\mu_\phi = i\omega\frac{1}{I_\phi}\left(\frac{N}{S}S' - \frac{\partial N}{\partial S}S' - \frac{\partial N}{\partial \rho}\rho'\right),$$

where $N = N_p + N_\phi$

By means of (3,1) and (3,2) the small increments S' and ρ', which are due to the presence of a sound wave, are expressed in terms of div $\underline{v}_n$ and div $\underline{v}_s$. Introducing these expressions into (6,11) gives

$$\mu_p = \frac{1}{\Gamma_\phi} \left\{ \frac{\partial N}{\partial \rho} \rho_s \, \text{div} \, v_s - \left(N - \frac{\partial N}{\partial S} S - \frac{\partial N}{\partial \rho} \rho_n \right) \text{div} \, v_n \right\} +$$

$$+ \frac{1}{\Gamma_{\phi\rho}} \left\{ \frac{\partial N_\rho}{\partial \rho} \rho_s \, \text{div} \, v_s - \left(N_\rho - \frac{\partial N_\rho}{\partial S} S - \frac{\partial N_\rho}{\partial \rho} \rho_n \right) \text{div} \, v_n \right\}, \qquad (6.12)$$

$$\mu_\phi = \frac{1}{\Gamma_\phi} \left\{ \frac{\partial N}{\partial \rho} \rho_s \, \text{div} \, v_s - \left(N - \frac{\partial N}{\partial S} S - \frac{\partial N}{\partial \rho} \rho_n \right) \text{div} \, v_n \right\}.$$

Comparison of the right hand sides of (6,5) and (6,6) with the above expressions, and the insertion of expressions (6,12) for μ_p and μ_ϕ, yield two equations

$$\zeta_n \, \text{div} \, v_n + \zeta_s \, \text{div} \, v_s =$$

$$\frac{1}{\Gamma_\phi} \left(\frac{\partial \rho}{\partial \mu_p} + \frac{\partial \rho}{\partial \mu_\phi} \right)_{\rho, S} \left\{ \left(N - \frac{\partial N}{\partial S} S - \frac{\partial N}{\partial \rho} \rho_n \right) \text{div} \, v_n - \frac{\partial N}{\partial \rho} \rho_s \, \text{div} \, v_s \right\} +$$

$$\frac{1}{\Gamma_{\phi\rho}} \left(\frac{\partial \rho}{\partial \mu_p} \right)_{\rho, S} \left\{ \left(N_\rho - \frac{\partial N_\rho}{\partial S} S - \frac{\partial N_\rho}{\partial \rho} \rho_n \right) \text{div} \, v_n - \frac{\partial N_\rho}{\partial \rho} \rho_s \, \text{div} \, v_s \right\}.$$

$$\zeta_n^S \, \text{div} \, v_n + \zeta_s^S \, \text{div} \, v_s =$$

$$\frac{\rho_s}{\Gamma_\phi} \left(\frac{\partial \Phi}{\partial \mu_p} + \frac{\partial \Phi}{\partial \mu_\phi} \right)_{\rho, S} \left\{ \left(N - \frac{\partial N}{\partial S} S - \frac{\partial N}{\partial \rho} \rho_n \right) \text{div} \, v_n - \frac{\partial N}{\partial \rho} \rho_s \, \text{div} \, v_s \right\} +$$

$$\frac{\rho_s}{\Gamma_{\phi\rho}} \left(\frac{\partial \Phi}{\partial \mu_p} \right)_{\rho, S} \left\{ \left(N_\rho - \frac{\partial N_\rho}{\partial S} S - \frac{\partial N_\rho}{\partial \rho} \rho_n \right) \text{div} \, v_n - \frac{\partial N_\rho}{\partial \rho} \rho_s \, \text{div} \, v_s \right\}.$$

Since these equations are supposed to hold for any

arbitrary values of $\underline{v}_n$ and $\underline{v}_s$, it follows that:

$$\zeta_n = \frac{1}{\Gamma_\phi}\left(\frac{\partial\rho}{\partial\mu_\rho} + \frac{\partial\rho}{\partial\mu_\phi}\right)_{\rho,S}\left(N - \frac{\partial N}{\partial S}S - \frac{\partial N}{\partial\rho}\rho_n\right) +$$

$$+ \frac{1}{\Gamma_{\phi\rho}}\left(\frac{\partial\rho}{\partial\mu_\phi}\right)_{\rho,S}\left(N_\rho - \frac{\partial N_\rho}{\partial S}S - \frac{\partial N_\rho}{\partial\rho}\rho_n\right),$$

$$\zeta_s = -\frac{1}{\Gamma_\phi}\left(\frac{\partial\rho}{\partial\mu_\rho} + \frac{\partial\rho}{\partial\mu_\phi}\right)_{\rho,S}\frac{\partial N}{\partial\rho}\rho_s - \frac{1}{\Gamma_{\phi\rho}}\left(\frac{\partial\rho}{\partial\mu_\phi}\right)_{\rho,S}\frac{\partial N_\rho}{\partial\rho}\rho_s \qquad (6.13)$$

$$\zeta_n^S = \frac{\rho_s}{\Gamma_\phi}\left(\frac{\partial\overline{\Phi}}{\partial\mu_\rho} + \frac{\partial\overline{\Phi}}{\partial\mu_\phi}\right)_{\rho,S}\left(N - \frac{\partial N}{\partial S}S - \frac{\partial N}{\partial\rho}\rho_n\right) +$$

$$+ \frac{\rho_s}{\Gamma_{\phi\rho}}\left(\frac{\partial\overline{\Phi}}{\partial\mu_\phi}\right)_{\rho,S}\left(N_\rho - \frac{\partial N_\rho}{\partial S}S - \frac{\partial N_\rho}{\partial\rho}\rho_n\right) \qquad (6.14)$$

$$\zeta_s^S = -\frac{\rho_s}{\Gamma_\phi}\left(\frac{\partial\overline{\Phi}}{\partial\mu_\rho} + \frac{\partial\overline{\Phi}}{\partial\mu_\phi}\right)_{\rho,S}\frac{\partial N}{\partial\rho}\rho_s - \frac{\rho_s}{\Gamma_{\phi\rho}}\left(\frac{\partial\overline{\Phi}}{\partial\mu_\rho}\right)_{\rho,S}\frac{\partial N_\rho}{\partial\rho}\rho_s$$

It would not be too difficult to evaluate these expres-

sions. We are, however, not going to proceed in this way,

since experiments on sound absorption are not sufficient

for determining all coefficients of viscosity. Only two

combinations of these coefficients can be derived from the

absorption coefficients of first and second sound. It is easy

to verify by means of (6,13) and (6,14) that the coefficients

$\zeta_n^S, \zeta_s^S, \zeta_s$, satisfy in fact the first of the relations (6,10).

We have still to express the absorption coefficients of

first and second sound in terms of the coefficients of first

and second viscosity. For that purpose the complete hydro-

dynamical equations, including viscosity terms, are written

out. According to (3,1), (3,2), (6,5) and (6,6).

$$\frac{\partial\rho}{\partial t} + \operatorname{div}\underline{j} = 0 \quad, \qquad \frac{\partial S}{\partial t} + S\operatorname{div}\underline{v}_n = 0$$

$$\frac{\partial\underline{j}}{\partial t} + \nabla\rho = \eta\,\Delta\underline{v}_n + \left(\zeta_n + \frac{1}{3}\eta\right)\nabla\operatorname{div}\underline{v}_n + \zeta_s\nabla\operatorname{div}\underline{v}_s \quad, \qquad (6.15)$$

$$\rho\frac{\partial\underline{v}_s}{\partial t} + \rho\nabla\overline{\Phi} = \zeta_n^S\nabla\operatorname{div}\underline{v}_n + \zeta_s^S\nabla\operatorname{div}\underline{v}_s$$

By eliminating from these equations $\underline{v}_n$, $\underline{v}_s$ and expressing all thermodynamic functions in terms of ρ and S, two simultaneous equations are obtained. The compatibility condition, similarly to (3,15), has the form of a determinant set equal to 0. Since the determinant would have a complicated form it will not be formulated explicitly. The roots of the determinantal equation are obtained by neglecting terms of the magnitude ω^2 and noting the relatively small magnitude of $\left(\partial_\rho/\partial S\right)_\rho$: —

$$u_1^2 = \left(\frac{\partial p}{\partial \rho}\right)_S + \left(\frac{\partial p}{\partial S}\right)_\rho \frac{S}{\rho} + \frac{i\omega}{\rho}\left(\frac{4}{3}\eta + \varsigma_n + \varsigma_s\right) \qquad (6.16)$$

$$u_2^2 = \frac{\rho_s}{\rho \rho_n} S^2 \left(\frac{\partial T}{\partial S}\right)_\rho + \frac{i\omega}{\rho_n}\left[\left(\frac{4}{3}\eta + \varsigma_n - \varsigma_s\frac{\rho_n}{\rho_s}\right)\frac{\rho_s}{\rho} - \left(\varsigma_n^s - \varsigma_s^s\frac{\rho_n}{\rho_s}\right)\right] \qquad (6.17)$$

From (6,16), (6,17) the absorption coefficients for 1st and 2nd sound are readily obtained:—

$$\alpha_1 = I_m\left(\frac{\omega}{u_1}\right) = \frac{\omega^2}{2\rho c^3}\left(\frac{4}{3}\eta + \varsigma_n + \varsigma_s\right) \qquad (6.18)$$

$$\alpha_2 = I_m\left(\frac{\omega}{u_2}\right) = \frac{\omega^2 \rho_s}{2\rho\rho_n u_{2o}^3}\left[\frac{4}{3}\eta + \left(\varsigma_n - \varsigma_s\frac{\rho_n}{\rho_s}\right) - \frac{\rho}{\rho_s}\left(\varsigma_n^s - \varsigma_s^s\frac{\rho_n}{\rho_s}\right)\right] \qquad (6.19)$$

By means of (6,13) and (6,14) it is easy to show that the terms in the absorption coefficient which depend upon the second viscosity are the same as expressions (4,3) and (5,1) respectively.

We find in fact from (6,13) and (6,14):

$$\varsigma_I \equiv \varsigma_n + \varsigma_s = \frac{1}{\Gamma_\phi}\left(\frac{\partial\rho}{\partial\mu_\phi} + \frac{\partial\rho}{\partial\mu_\phi}\right)_{\rho,S}^2 + \frac{1}{\Gamma_{\phi\rho}}\left(\frac{\partial\rho}{\partial\mu_\phi}\right)_{\rho,S}^2 \qquad (6.20)$$

$$\varsigma_{II} \equiv \left(\varsigma_n - \varsigma_s\frac{\rho_n}{\rho_s}\right) - \frac{\rho}{\rho_s}\left(\varsigma_n^s - \varsigma_s^s\frac{\rho_n}{\rho_s}\right) =$$

$$= \frac{1}{\Gamma_\phi}\left(\frac{\partial\rho}{\partial\mu_\phi} + \frac{\partial\rho}{\partial\mu_\phi} - \rho\frac{\partial\Phi}{\partial\mu_\phi} - \rho\frac{\partial\Phi}{\partial\mu_\phi}\right)_{\rho,S}^2 + \frac{1}{\Gamma_{\phi\rho}}\left(\frac{\partial\rho}{\partial\mu_\phi} - \rho\frac{\partial\Phi}{\partial\mu_\phi}\right)_{\rho,S}^2 \qquad (6.21)$$

In figure 2 the combination of coefficients of second viscosity determining the absorption of 1st and 2nd sound is plotted against temperature. For the purpose of comparison, the viscosity coefficient η is also shown in this figure.

It is clearly seen that the absorption of first sound is
mainly due to second viscosity. The sum S_I is at
all temperatures larger than the coefficient

It was pointed out before that the coefficients of first

and second viscosity are usually of comparable magnitude.

The marked prevlance of the effect of second viscosity with

respect to the absorption of first sound is explained by an

anomalously large value of the differential coefficient

$\partial c/\partial \rho$ on which the magnitude of $\frac{\rho}{I}$ depends essentially.

The absorption coefficient for second sound shows a

different relation. Here the coefficient $\frac{\rho}{II}$ is of similar

magnitude as at all temperatures. The absorption of second

sound is accordingly determined to equal parts by the first

and second viscosity.

If absolute zero is approached the coefficients $\frac{\rho}{I}$ and

S_{II} increase rapidly. By means of (6,20) (6,21), (4,17)

and (4,16) it is possible to obtain their limiting values at

extremely low temperatures.

$$\frac{\rho}{I} = \frac{N_\rho^2}{\Gamma_{\phi\rho}}\left(\frac{\Delta}{3T}\right)^2\left(1 + \frac{3\rho}{c}\frac{\partial c}{\partial \rho}\right)^2 \approx \frac{1.25 \cdot 10^{-3}}{T}\left(1 + \frac{3\rho}{c}\frac{\partial c}{\partial \rho}\right)^2,$$

$$(6.22)$$

$$\frac{\rho}{II} = \frac{N_\rho^2}{\Gamma_{\phi\rho}}\left(\frac{\Delta}{3T}\right)^2 \approx \frac{1.25 \cdot 10^{-3}}{T}$$

It is seen that in the vicinity of absolute zero the

effect of the second viscosity on the absorption of first sound

is $\left[1 + \left(\frac{3\rho}{c}\right)\left(\frac{\partial c}{\partial \rho}\right)\right]^2$ times larger than the corresponding effect for
second sound.

Comparison of (6,18) and (6,19) shows that owing to the

factor ρ_s/ρ_n in (6,19), the absorption coefficient of second

sound, α_2, increases at a higher rate than the absorption

coefficient of first sound, when the temperature is decreased.

Near absolute zero the absorption coefficient α_2 is markedly

higher than α_1.

In figure 3 the ratio α_1/ω^2 and the ratio α_2/ω^2 are plotted against the temperature. In applying this diagram it must be remembered that the coefficients of absorption are proportional to α only for those frequencies for which condition (4,1) is valid. On the same diagram absorption coefficients for first sound are given corresponding to a frequency of $1.5 \cdot 10^7 \ sec^{-1}$ in a range of temperatures for which experimental data are available. Here the theoretical graph as shown is a complicated curve; the points shown are experimental values [1].

In calculating the values of α_1/ω^2 and α_2/ω^2 the slight variation of the density with temperature is neglected; the density of helium II is assumed to be constant and equal to $0.145 \ gram/cm^3$.

In conclusion we should like to make some remarks concerning the absorption of sound near the λ-point. The above relations are obtained on the assumption that phonons and rotons are forming an ideal gas. It has been pointed out earlier [3] that this assumption is no longer valid near the λ-point.

The effect of the temperature on the absorption coefficient cannot be determined in a simple way by means of the above arguments. Nevertheless it should be admissible to draw at least a qualitative conclusion from (6,19): the absorption coefficient of second sound is found to increase at a high rate if the λ-point is approached. The velocity of second sound approaches zero if the λ-point is approached; the same holds for the ratio ρ_s/ρ_n.

The factor $\rho_s/\rho_n \ u_{20}^3$ appearing in (6,19) as a factor of the bracket, will, in approaching the λ-point become infinite of the order $1/u_{20}$ on account of equation (3,29). As the bracket itself in (6,19) does not vanish, the absorption coefficient of second sound will increase somewhat like $1/u_{20}$ if the λ-point is approached. Experimental data [7] concerning the absorption of second sound are inconclusive.

The only inference they admit is the statement that absorption of second sound increases appreciably if the λ-point is approached.

I should like to express my sincere thanks to Academician L.D. Landau for continual advice and to Professor E.M. Lifshitz for helpful discussion.

Appendix

The derivatives $\left(\partial N_\rho / \partial \mu_\rho\right)_{\rho,S}$ and $\left(\partial N_\phi / \partial \mu_\phi\right)_{\rho,S}$ are conveniently calculated in terms of the independent variables ρ and T. Transformation is carried out by means of the Jacobians:

$$\left(\frac{\partial N_\rho}{\partial \mu_\rho}\right)_S = \frac{\partial(N_\rho,S)}{\partial(\mu_\rho,S)} = \frac{\partial(N_\rho,S)}{\partial(\mu_\rho,T)} \Big/ \frac{\partial(\mu_\rho,S)}{\partial(\mu_\rho,T)}$$
$$= \left(\frac{\partial N_\rho}{\partial \mu_\rho}\right)_T - \left(\frac{\partial N_\rho}{\partial T}\right)_{\mu_\rho}\left(\frac{\partial S}{\partial \mu_\rho}\right)_T \Big/ \left(\frac{\partial S}{\partial T}\right)_{\mu_\rho} \qquad (A1)$$

From (3.20) we obtain (where ρ is held constant)
$$\left(\partial N_\rho / \partial T\right)_{\mu_\rho} = \left(\partial S / \partial \mu_\rho\right)_T,$$
and transform (A.1) into the form

$$\left(\frac{\partial N_\rho}{\partial \mu_\rho}\right)_S = \left(\frac{\partial N_\rho}{\partial \mu_\rho}\right)_T - \left(\frac{\partial N_\rho}{\partial T}\right)_{\mu_\rho}^2 \Big/ \left(\frac{\partial S}{\partial T}\right)_{\mu_\rho} \qquad (A.2)$$

Similarly we find

$$\left(\frac{\partial N_\phi}{\partial \mu_\phi}\right)_S = \left(\frac{\partial N_\phi}{\partial \mu_\phi}\right)_T - \left(\frac{\partial N_\phi}{\partial T}\right)_{\mu_\phi}^2 \Big/ \left(\frac{\partial S}{\partial T}\right)_{\mu_\phi} \qquad (A.3)$$

Since the distribution function for rotons has to be in accordance with Boltzmann statistics, it is readily found that $\left(\partial N_\rho / \partial \mu_\rho\right)_{\rho,T}$ is equal to N_p/kT.

The total number of phonons per unit volume of the liquid in a state of non-equilibrium specified by the chemical potential μ_ϕ is equal to

$$N_\phi = \frac{1}{(2\pi\hbar)^3} \int_0^\infty \left[e^{(c\rho - \mu_\phi)/kT} - 1 \right]^{-1} 4\pi\rho^2 d\rho \qquad (A.4)$$

Differentiation of (A.4) with respect to μ_ϕ at constant T and subsequent partial integration gives

$$\left(\frac{\partial N_\phi}{\partial \mu_\phi}\right)_{\rho,T} \approx \frac{N_\phi}{kT} \frac{5\pi^2}{36} \qquad (A.5)$$

The remaining differential coefficients appropriate to equilibrium are obtained by means of (4,8)-(4,10):

$$\left(\frac{\partial N_\rho}{\partial T}\right)_{\rho,\,\mu_\rho} = \frac{N_\rho}{T}\left(\frac{1}{2} + \frac{\Delta}{T}\right), \quad \left(\frac{\partial N_\phi}{\partial T}\right)_{\rho,\,\mu_\phi} = \frac{3\,N_\phi}{T}$$

$$\left(\frac{\partial S}{\partial T}\right)_{\mu_\rho,\,\mu_\phi,\,\rho} = \frac{k}{T}\left[N_\rho\left(\frac{\Delta^2}{T^2} + \frac{\Delta}{T} + \frac{3}{4}\right) + N_\phi\frac{\pi^4}{9}\right] \qquad (A6)$$

By means of (A,4) - (A,6) the final expressions as given in section 4, i.e. (4,6), (4,7), are derived from (A,2), (A,3).

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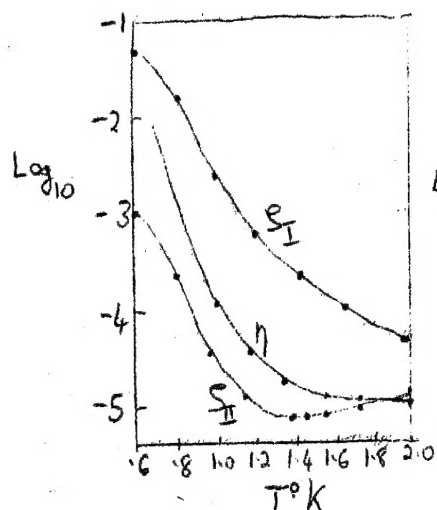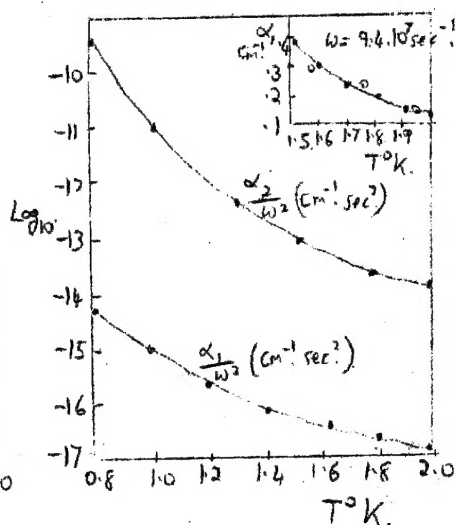7. J. Pellam, Phys. Rev. 75, 1183, 1949.

Fig. 2

Fig. 3